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MATHEMATICAL PHYSICS

Force Acting on a Rough Disk Spinning in a Flow of Noninteracting Particles

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1. Consider a flow of point particles impinging on a body spinning around a fixed point. The particles do not interact with one another, and their collisions with the body are elastic. The goal is to determine the pressure force exerted by the flow on the body.

The problem is considered in two dimensions. In the Euclidean space $\mathbb{R}^2$, we introduce an orthonormal frame of reference Ox_1x_2. The flux density ρ is a constant. Initially, the particles move at the identical velocity $\vec{v} = (0; -v)^T$. Here and below, the vectors are represented as columns. The body is a rough disk, i.e., a set that differs slightly (in some sense) from a disk. Specifically, the rough disk is understood as follows. Consider a sequence of sets B_n, $n = 3, 4, \ldots$ contained in the disk $B_r(O)$ of radius r centered at the origin O. The set B_n is invariant under the rotation about O through an angle of $\frac{2\pi}{n}$, and the intersection of its boundary ∂B_n with some sector $OA_{1n}A_{2n}$ of angular size $\frac{2\pi}{n}$ is a piecewise smooth non-self-intersecting curve I_n contained in the triangle $OA_{1n}A_{2n}$ with the ends A_{1n} and A_{2n}. Finally, it is assumed that all the curves I_n are similar to each other; i.e., they can be superimposed on one another via isometry and similarity transformations.

The rough disk $\mathcal{B}$ (of radius r centered at O) is identified with this sequence of sets B_n. Thus, the sets B_n can be viewed as successive approximations of the rough disk. They approximate the disk $B_r(O)$ and have a boundary with a similar structure.

The body spins counterclockwise about O at a constant angular velocity $\omega > 0$. The initial position of the body (at $t = 0$) coincides with the set B_n (see Fig. 1). The pressure force exerted by the flow on the body is a $\frac{2\pi}{n\omega}$-periodic vector function $\vec{F}_{B_n,\omega}(t)$. As $n \to \infty$, it tends to a constant vector $\vec{F}(\mathcal{B}, \omega)$, which is called the pressure force exerted on the rough disk. Of course, this quantity depends not only on ω but also on the choice of a rough body, i.e., on a particular sequence B_n. The problem is

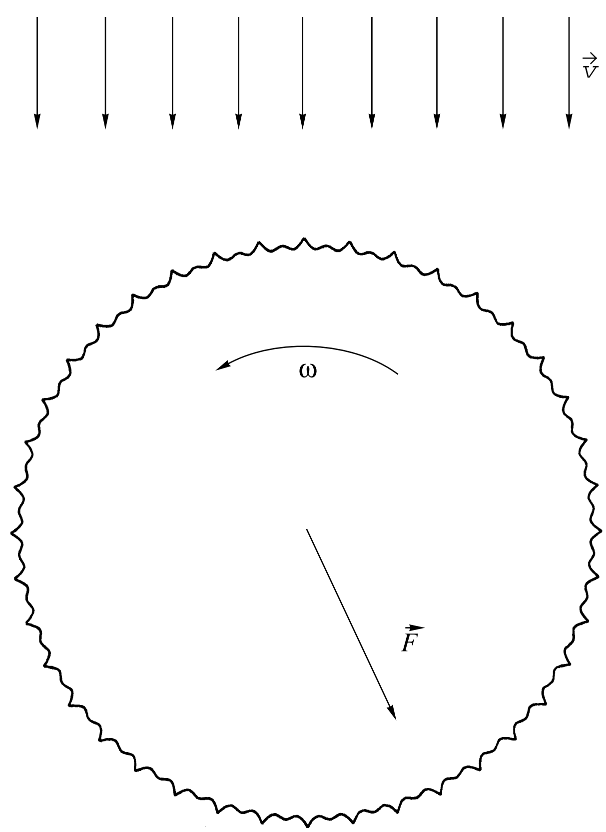

Fig. 1. Spinning disk in the flow of particles.

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to derive analytical formulas that, given ω, yield the set of all possible vectors $\vec{F}(\mathcal{B}, \omega)$. Additionally, we construct this set numerically for some values of ω. The convergence mentioned above is not proved rigorously. The passage to the limit in Section 3 is heuristic, i.e., the pressure force is calculated for large n.

2. For a fixed n, we consider the set $B = B_n$. Let $\xi \in \partial(\mathrm{conv}B)$. Let $\vec{n}_\xi \in S^1$ denote the outward normal to $\partial(\mathrm{conv}B)$ at the point ξ. The angle between $\vec{n}_\xi$ (or $-\vec{n}_\xi$) and another vector is measured counterclockwise from $\vec{n}_\xi$ (or $-\vec{n}_\xi$) to the given vector. Denote by $|\partial(\mathrm{conv}B)|$ the length of the curve $\partial(\mathrm{conv}B)$. The probability measure μ on $\partial(\mathrm{conv}B) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $\partial\mu(\xi, \varphi) = \frac{1}{2|\partial(\mathrm{conv}B)|}\cos\varphi\, d\xi\, d\varphi$, where $d\xi$ and $d\varphi$ are the line elements on $\partial(\mathrm{conv}B)$ and S^1, respectively. Consider a billiard in $\mathbb{R}^2 \backslash B$ and define the mapping $(\xi, \varphi) \mapsto (\xi_B^+(\xi, \varphi), \varphi_B^+(\xi, \varphi))$ as follows. Suppose that a billiard particle intersects $\partial(\mathrm{conv}B)$ (from the outside inward) at the point ξ and its velocity at that moment makes an angle of φ with the vector $-\vec{n}_\xi$. Then, after several reflections from ∂B, the particle intersects $\partial(\mathrm{conv}B)$ the second time (from inside the disk outward) at the point $\xi_B^+(\xi, \varphi)$ and its velocity at that moment makes an angle of $\varphi_B^+(\xi, \varphi)$ with $\vec{n}_\xi$. If $\xi \in \partial(\mathrm{conv}B) \cap \partial B$, then we set $\xi_B^+(\xi, \varphi) = \xi$ and $\varphi_B^+(\xi, \varphi) = -\varphi$.

The mapping $\xi_B^+(\xi, \varphi)$ and $\varphi_B^+(\xi, \varphi)$ is a one-to-one map of a full-measure subset of $\partial(\mathrm{conv}B) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ into itself; moreover, it is involutive and preserves the measure μ. The probability measure ν_B on $\square := \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $\nu_B(A) = \mu(\{(\xi, \varphi): (\varphi, \varphi_B^+(\xi, \varphi)) \in A\})$ for any Borel set $A \subset \square$. In addition to B, the measure ν_B is an important characteristic of billiard scattering. It describes the joint distribution of the pair (φ, φ^+) (the angle of incidence and the angle of reflection) for a randomly chosen particle reflected from B.

By the choice of the sequence B_n, ν_{B_n} is independent of n. Therefore, $\nu_{B_n} =: \nu_{\mathcal{B}}$ is well defined. In fact, this measure defines the law of billiard scattering on the rough set $\mathcal{B}$.

The measure λ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $d\lambda(\varphi) = \frac{1}{2}\cos\varphi d\varphi$. Let $\mathcal{M}$ denote the set of measures ν on $\square$ with coordinates φ and φ^+ such that (a) both projections of ν onto the φ and φ^+ axes coincide with λ; and (b) ν is invariant under the mapping $(\varphi, \varphi^+) \mapsto (\varphi^+, \varphi)$.

Theorem. $\mathrm{cl}\{\nu_{\mathcal{B}}$: *$\mathcal{B}$ is a rough disk of radius r*$\} = \mathcal{M}$ *for any $r > 0$. Here,* cl *denotes the closure of the set of measures in the weak convergence topology.*

This theorem is deduced by a small modification of the proof of Theorem 1 in [3].

3. Consider a particle of the flow that intersects $\partial(\mathrm{conv}B)$ at some (say, zero) time at a point ξ and the vector $\vec{n}_\xi$ makes an angle of $\varphi = \varphi(\xi)$ with the vertical axis: $\vec{n}_\xi = (-\sin\varphi; \cos\varphi)^T$, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We pass to the frame of reference $\tilde{O}\tilde{x}_1\tilde{x}_2$ whose origin is at ξ (and rotates together with the latter), the $\tilde{O}\tilde{x}_1$ axis is aligned with the vector $\vec{e}_\varphi := (\cos\varphi; \sin\varphi)^T$, and the $\tilde{O}\tilde{x}_2$ axis is aligned with the vector $\vec{n}_\xi$. For vectors issuing from the point ξ at the zero moment, the coordinates change according to the following rule. Let the velocity has the form $\vec{u} = (u_1; u_2)^T$ in the original frame and $\vec{\tilde{u}} = (\tilde{u}_1; \tilde{u}_2)^T$ in the moving frame. Then $\vec{\tilde{u}} = A_{-\varphi}\vec{u} + r\omega\vec{e}_0$ and $\vec{u} = A_\varphi\vec{\tilde{u}} - r\omega\vec{e}_\varphi$, where $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$.

For large n and $B = B_n$, the sojourn time of a particle in $\mathrm{conv}B$ is little and we can approximately assume that the frame $\tilde{O}\tilde{x}_1\tilde{x}_2$ moves rectilinearly and uniformly over this time. At the first-passage time for $\partial(\mathrm{conv}B)$ in the moving frame, the velocity $\vec{v} = (0; -v)^T$ becomes $\vec{\tilde{v}} = (r\omega - v\sin\varphi; -v\cos\varphi)^T = \zeta(\sin x; -\cos x)^T$, where

$$\begin{aligned} \zeta &= \zeta(\varphi) = \sqrt{r^2\omega^2 - 2r\omega v\sin\varphi + v^2}, \\ x &= x(\varphi) = \arcsin\frac{r\omega - v\sin\varphi}{\zeta(\varphi)}. \end{aligned} \tag{1}$$

At the second-passage time for $\partial(\mathrm{conv}B)$ in the moving frame, the velocity is also equal to ζ in magnitude and makes an angle of $y = \varphi_B^+(\xi, x)$ with the normal $\vec{n}_\xi$; i.e., $\vec{\tilde{v}}^+ = \zeta(-\sin y; \cos y)^T$. In the original frame, this velocity has the form $\vec{v}^+ = \vec{v}^+(\xi, \varphi) = (-\zeta\sin(\varphi + y) - r\omega\cos\varphi; \zeta\cos(\varphi + y) - r\omega\sin\varphi)^T$.

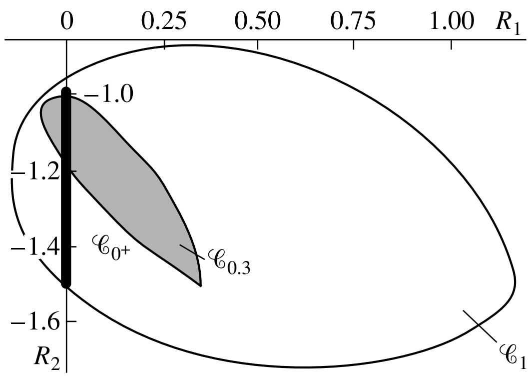

Fig. 2. Convex sets are shown: the segment $\mathscr{C}_{0^+}$ with the endpoints (0, –0.9878…) and (0, –1.5), $\mathscr{C}_{0.3}$ (shaded), and $\mathscr{C}_1$.

The pressure force $\vec{F}$ is calculated by integrating with respect to ξ $\rho v(\vec{v} - \vec{v}^{\,+})\cos\varphi$: $\vec{F} = \vec{F}_{B,\,\omega}(0) = \rho v \int\limits_{\partial(\mathrm{conv}B)} (\vec{v} - \vec{v}^{\,+}(\xi, \varphi(\xi)))(\cos\varphi(\xi))_+ d\xi$, where $z_+ = \max\{z, 0\}$ is the positive part of z. When ξ rotates through an angle of $\frac{2\pi}{n}$, $\varphi(\xi)$ increases by $\frac{2\pi}{n}$; i.e., $\varphi(A_{2\pi/n}\xi) = \varphi(\xi) + \frac{2\pi}{n}$. Therefore, passing to the limit as $n \to \infty$ yields

$$\vec{F}(\mathscr{B}, \omega)$$

$$= 2r\rho v \int\limits_{\partial(\mathrm{conv}B)\times[-\pi/2,\,\pi/2]}\int (\vec{v} - \vec{v}^{\,+}(\xi, \varphi))\,d\mu(\xi, \varphi).$$

Finally, making the substitution $(\xi, \varphi) \mapsto (x, y)$ and using formulas (1), after rather cumbersome computations, we obtain

$$\vec{F}(\mathscr{B}, \omega) = \frac{8}{3}\rho r v^2 \cdot \vec{R}\left(\nu_{\mathscr{B}}, \frac{\omega r}{v}\right),$$
$$\vec{R}(\nu, \lambda) = \iint\limits_{\square} \vec{c}(x, y, \lambda)\,d\nu(x, y). \tag{2}$$

The integrand is

$$\vec{c}(x, y, \lambda) = \frac{3}{2}\frac{(\lambda\sin x + \sin\eta)^3}{\sin\eta}$$
$$\times \cos\frac{x-y}{2}\begin{pmatrix} \cos\left(\eta + \frac{x-y}{2}\right) \\ -\sin\left(\eta + \frac{x-y}{2}\right) \end{pmatrix}$$

for $0 < \lambda \le 1$ and

$$\vec{c}(x, y, \lambda) = \frac{3\cos\frac{x-y}{2}}{\sin\eta}\{(\lambda^3\sin^3 x + 3\lambda\sin x\sin^2\eta)$$
$$\times \cos\eta\left[\cos\frac{x-y}{2}; -\sin\frac{x-y}{2}\right]^T$$
$$- (3\lambda^2\sin^2 x\sin\eta + \sin^3\eta)$$
$$\times \sin\eta\left[\sin\frac{x-y}{2}; \cos\frac{x-y}{2}\right]^T \chi_{x \ge x_0}(x, y) \quad \text{for} \quad \lambda > 1.$$

Here, $\eta = \eta(x, \lambda) = \arcsin\sqrt{1 - \lambda^2\cos^2 x}$, χ denotes the characteristic function, and $x_0 = x_0(\lambda) = \arccos\frac{1}{\lambda}$. Specifically,

$$\vec{c}(x, y, 1) = 6\sin^2 x\begin{pmatrix} \cos(2x - y) + \cos x \\ -\sin(2x - y) - \sin x \end{pmatrix}\chi_{x \ge 0}(x, y)$$

and $\vec{c}\,(x, y, 0^+) = \frac{3}{4}(1 + \cos(x - y))(0; -1)^T$. The coefficients multiplying $\vec{R}$ are chosen so that $\vec{R}\,(\nu_0, \lambda) = (0; -1)^T$ for the measure ν_0 with the density $\frac{1}{2}\cos\varphi\delta(\varphi + \varphi^+)$, which corresponds to elastic reflection from a convex body. Accordingly, for the disk $B_r(O)$, we have $\vec{F}\,(B_r(O), \omega) = \frac{8}{3}\rho r v^2(0; -1)^T$, which is supported by direct computations. Thus, as expected, the pressure force exerted on the disk is independent of its angular velocity is parallel to the direction of the flow.

The vector-valued Monge–Kantorovich problem, which is to find the set $\mathscr{C}_\lambda = \{\vec{R}\,(\nu, \lambda)\colon \nu \in \mathscr{M}\}$, was solved numerically for $\lambda = 0^+$, 0.3, and 1. The results are shown in Fig. 2.

Another important characteristic of the interaction is the moment of the pressure force, which slows down the rotation. It is equal to

$$-2r^2\rho v \int\limits_{\partial(\mathrm{conv}B)\times[-\pi/2,\,\pi/2]}\int \langle \vec{v} - \vec{v}^{\,+}(\xi, \varphi), e_\varphi\rangle\,d\mu(\xi, \varphi).$$

Computations give $-\frac{8}{3}\rho r^2 v^2 \cdot R_M\left(\nu_{\mathscr{B}}, \frac{\omega r}{v}\right)$, where

$$R_M(\nu, \lambda) = \iint\limits_{\square} c_M(x, y, \lambda)\,d\nu(x, y),$$

$$c_M(x, y, \lambda) = \begin{cases} \dfrac{3(\lambda\sin x + \sin\eta)^3(\sin x + \sin y)}{4\sin\eta} & \text{if } 0 < \lambda \le 1 \\ \dfrac{3(\lambda^3\sin^3 x + 3\lambda\sin x\sin^2\eta)}{2\sin\eta}(\sin x + \sin y)\chi_{x \ge x_0}(x, y) & \\ \text{if } \lambda > 1. & \end{cases}$$

The functional R_M is nonnegative and reaches its least value at ν_0: $R_M(\nu_0) = 0$. Thus, for $B_r(O)$, the moment is zero. For $0 < \lambda \le 1$, the largest value of the functional is $R_M(\nu_\star) = 1.5\lambda$, where $\nu_\star$ is a measure with the density $\frac{1}{2}\cos\varphi\delta(\varphi - \varphi^+)$.

4. Now, consider a three-dimensional body spinning about a fixed axis in a gas flow. If the pressure force is not parallel to its velocity, then the Magnus effect takes place. If the transverse force is aligned with the instantaneous velocity of the front point (facing the flow) of the body, then we have the proper Magnus effect. If the transverse force is opposite to this velocity, then the reverse Magnus effect occurs.

The Magnus effect has been addressed in numerous publications. In [4, 2, 1, 5], it is studied in the case of a gas that is so rarefied that the process can be described in terms of free molecular flow. It is assumed that the body is convex and symmetric about the axis of rotation (a ball or a cylinder was considered more frequently). The interaction of the gas particles with the body is assumed to be inelastic. Part of the tangential momentum of the particles is transferred to the body, which gives rise to a transverse force. The basic conclusion drawn in [4, 2, 1, 5] is that the reverse Magnus effect is observed in the given conditions. The formula for the transverse force derived in [4, 2, 1, 5] for the case when the velocity flow is perpendicular to the rotation axis (usually under certain additional assumptions) becomes

$$F_T = \frac{1}{2}\alpha M v\omega. \tag{3}$$

Here, M is the mass of the gas displaced by the body and the coefficient α ranges from 0 to 1 and is a measure of inelasticity in the interactions of the particles with the body. Additionally, the moment of the pressure force was calculated in [2].

In our opinion, the reverse Magnus effect is strongly rarefied media can be caused by the combined effect of the following two factors: (i) inelastic collisions and (ii) multiple collisions of particles with the body, which arise when the surface of the body is not convex but contains small cavities.

The influence of factor (i) was studied in [4, 2, 1, 5]. We analyzed the effect of factor (ii). Assuming that the collisions of the particles with the body are elastic, we excluded factor (i) from consideration. The two-dimensional case was considered in our simplified model. The internal temperature of the flow was assumed to be zero. We believe that the approach described can be extended to the three-dimensional case and media with a nonzero temperature.

By using (2), the transverse pressure force $F_1 = F_1(\mathcal{B}, \omega)$ in our model can be written as

$$F_1 = \frac{8}{3\pi}Mv\omega\frac{R_1(\nu_{\mathcal{B}}, \lambda)}{\lambda}. \tag{4}$$

Here, $\lambda = \frac{\omega r}{v}$ is the relative rotational velocity of the body, $M = \pi r^2\rho$ is the total mass of the particles displaced by the body, and R_1 is the first component of $\vec{R}$. Formula (4) resembles formula (3). Specifically, $\frac{R_1(\nu_{\mathcal{B}}, \lambda)}{\lambda}$ is similar to the coefficient α in (3). However, it is not a constant but depends in a rather complicated manner on the type of roughness of the body. Preliminary numerical experiments for particular rough disks suggest that this quantity varies weakly. Figure 2 shows that R_1 is positive for most solutions. Therefore, the reverse Magnus effect takes place. Specifically, for $\lambda = 1$, we conclude that 93.6% of the area of the corresponding shape is in the right half-plane $R_1 > 0$.

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REFERENCES

1. K. I. Borg, L. H. Söderholm, and H. Essén, Phys. Fluids **15**, 736–741 (2003).
2. S. G. Ivanov and A. M. Yanshin, Fluid Dyn. **15**, 449 (1980).
3. A. Plakhov, Arch. Ration. Mech. Anal. (in press).
4. C.-T. Wang, AIAA J. **10**, 713 (1972).
5. P. D. Weidman and A. Herczynski, Phys. Fluids **16**, L9–L12 (2004).